TUTORIAL - 3

I. Derive MGF, Mean, variance of

i) Binomial distribution.

$$P(X=x) = \cancel{\#} \, nC_x \, p^x q^{n-x}, \quad x = 0,1,2,3,\ldots n.$$

MGF:

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} P(x)$$

$$= \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$$

$$= nC_0 (pe^t)^0 q^n + nC_1 (pe^t)^1 q^{n-1} + \ldots + nC_n$$

$$M_X(t) = (pe^t + q)^n$$

Mean $E(x)$ $=$ $[M_X'(t)]_{t=0}$

$$= \left[\frac{d}{dt}(pe^t+q)^n\right]_{t=0}$$

$$= \left[n(pe^t+q)^{n-1} pe^t\right]_{t=0} \qquad \because p+q=1$$

$$= n(p+q)^{n-1} p \qquad (p+q)^{n-1} = (1)^{n-1} = 1$$

Mean $= np$.

$$E(x^2) = [M_X''(t)]_{t=0} = \left[\frac{d}{dt} npe^t (pe^t+q)^{n-1}\right]_{t=0}$$

$$= np\left[\frac{d}{dt} \underbrace{e^t (pe^t+q)^{n-1}}_{uv}\right]_{t=0}$$

$$= np\left[e^t (n-1) \underbrace{(pe^t+q)}_{1}^{n-2} pe^t + \underbrace{(pe^t+q)}_{1}^{n-1} e^t\right]_{t=0}$$

$$= np[(n-1)p + 1] = np(np - p + 1)$$

$$E(x^2) = n^2p^2 - np^2 + np$$

$$\text{Variance} = E(x^2) - (E(x))^2 = n^2p^2 - np^2 + np - (np)^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2} \qquad \because p+q=1, \quad q = 1-p$$

$$= np(1-p)$$

Variance $= npq$

ii) Poisson distribution :

$$P[X=x] = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

MGF:

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^{x}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots\right] = e^{-\lambda} e^{\lambda e^t}$$

$$M_X(t) = e^{\lambda(-1+e^t)}$$

Mean:

$$E(x) = [M_X'(t)]_{t=0} = \left[\frac{d}{dt} e^{-\lambda} e^{\lambda e^t}\right]_{t=0}$$

$$= e^{-\lambda}\left[\frac{d}{dt} e^{\lambda e^t}\right]_{t=0} = e^{-\lambda}\left(e^{\lambda e^t} \cdot \lambda e^t\right)_{t=0}$$

$$= e^{-\lambda} e^{\lambda} \cdot \lambda$$

Mean $= \lambda$

$$E(x^2(t)) = [M_X''(t)]_{t=0} = e^{-\lambda}\left[\frac{d}{dt}(\lambda e^t \cdot e^{\lambda e^t})\right]_{t=0}$$

$$= \lambda e^{-\lambda}\left[\frac{d}{dt}(\underset{u}{e^t} \cdot \underset{v}{e^{\lambda e^t}})\right]_{t=0}$$

$$= \lambda e^{-\lambda} \cdot \left[e^t e^{\lambda e^t} \lambda e^t + e^{\lambda e^t} \cdot e^t\right]_{t=0}$$

$$= \lambda e^{-\lambda}(\lambda e^{\lambda} + e^{\lambda})$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda e^{-\lambda} e^{t}$$

$$E(x^2(t)) = \lambda^2 + \lambda$$

$$\text{Variance} = E(x^2) - (E(x))^2$$

$$= \lambda^2 + \lambda - (\lambda)^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

Variance $= \lambda$.

1. Out of 500 families 4 children each. How many families would be expect to have,

Given $n = 4$

$N = 500$

$p = \frac{1}{2}, \quad q = \frac{1}{2}.$

By binomial distribution

$$P(X = x) = nC_x \, p^x q^{n-x}$$

$$= 4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$$

$$= 4C_x \left(\frac{1}{2}\right)^4$$

out of 500 families

$$P(X = x) = N \, P(X = x)$$

$$= (500) \, 4C_x \left(\frac{1}{2}\right)^4$$

i) 2 boys and 2 girls

$$P[\text{2 boys and 2 girls}] = P[X = 2]$$

$$= 4C_2 \left(\frac{1}{2}\right)^4$$

$$= 0.875$$

out of 500 families $P[X = 2] = (500) \times 0.375$

$= 187.5 \quad \sim 187$ families

ii) atleast one boy.

$$P[\text{atleast one boy}] = P[X \geq 1]$$

$$= 1 - P[X < 1]$$

$$= 1 - P[X = 0]$$

$$= 1 - 4C_0 \left(\frac{1}{2}\right)^4$$

$$= 1 - 0.0625$$

$$= 0.9375$$

out of 500 families $P[X \geq 1] = 500 \times 0.9375$

$= 468.75$

~ 468 families

iii) atmost 2 girls

$P[\text{atmost 2 girls}] = P[x \le 2]$

$= P[x=0] + P[x=1] + P[x=2]$

$= 4C_0\left(\frac{1}{2}\right)^4 + 4C_1\left(\frac{1}{2}\right)^4 + 4C_2\left(\frac{1}{2}\right)^4$

$= 0.6875$

out of 500 families $= 500 \times 0.6875$

$= 343.75$

~ 343 families

iv) childrens of both gender.

$P[\text{childrens of both genders}] = 1 - P[\text{same genders}]$

$= 1 - (P[x=0] + P[x=4])$

$= 1 - \left[{}^4C_0\left(\frac{1}{2}\right)^4 + {}^4C_4\left(\frac{1}{2}\right)^4\right]$

$= 1 - (0.0625 + 0.0625)$

$= 1 - 0.125$

$P[x=1] + P[x=2] + P[x=3] = 0.875$

out of 500 families $= 500 \times 0.875$

$= 437.5$

~ 437 families

2. A manufacture who produces medicine bottles finds out 0.1% of bottles are defective and the bottles are packed in boxes contains 500 bottles each. A drug manufacture who buys 100 boxes from the producer, use poisson distribution how many boxes will contain

Sol: $P[\text{bottles are defective}] = 0.1\% = p$

$0.1\% = \frac{0.1}{100}$

$= 0.001$

$p = 0.1$

$N = 100$

$n = 500$

Given $n = 500$

$N = 100$

w.k.t. $\lambda = np$

$\lambda = 500 \times 0.001$

$\lambda = 0.5$

For Poisson distribution of pdf is

$$P[X=x] = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-0.5}(0.5)^{x}}{x!}$$

i) no defective bottles

$$P[\text{no defective bottles}] = P[X=0]$$

$$= \frac{e^{-0.5}(0.5)^{0}}{0!}$$

$$= e^{-0.5}$$

$$= 0.6065$$

Out of 100 boxes $P[X=0] = 100 \times 0.6065$

$$= 60.65$$

$$\sim 60 \text{ boxes.}$$

ii) atleast 1 defective bottles

$$P[\text{atleast 1 defective bottles}] = P[X \geq 1]$$

$$= 1 - P[X<1]$$

$$= 1 - P[X=0]$$

$$= 1 - 0.6065$$

$$= 0.3935$$

Out of 100 boxes $P[X \geq 1] = 100 \times 0.3935$

$$= 39.35 \sim 39 \text{ boxes.}$$

IX Derive MGF, Mean, variance of

iii) Uniform Distribution

PDF of uniform distribution is given by

$$f(x) = \frac{1}{b-a}, \quad a < x < b$$

$$MGF = M_X(t) = E(e^{tx}) = \int e^{tx} f(x)\,dx$$

$$= \int_a^b e^{tx} \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b e^{tx}\,dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b = \frac{1}{b-a}\left[\frac{e^{bt}-e^{at}}{t}\right]$$

$$M_x(t) = \frac{e^{bt}-e^{at}}{t(b-a)}$$

Mean = $E(x) = \int x\, f(x)\, dx = \int_a^b \frac{1}{b-a}\, x\, dx$

$$= \frac{1}{b-a}\int_a^b x\, dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{b^2-a^2}{2(b-a)}$$

$$= \frac{(b+a)(b-a)}{2(b-a)}$$

Mean $= \frac{b+a}{2}$ or $\frac{a+b}{2}$

$$E(x^2) = \int x^2 f(x)\, dx = \int_a^b \frac{1}{b-a}\, x^2\, dx$$

$$= \frac{1}{b-a}\int_a^b x^2\, dx$$

$$= \frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)}$$

$$E(x^2) = \frac{b^2+ab+a^2}{3}$$

Variance $= E(x^2) - (E(x))^2$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{b^2+2ab+a^2}{4}\right)$$

$$= \cancel{\frac{b^2+ab+a^2}{3}} = \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

Variance $= \frac{(b-a)^2}{12}$ or $\frac{(a-b)^2}{12}$

(iv) Exponential distribution

The pdf of exponential distribution is

$$\cancel{f(x) = \alpha e^{-\alpha x}}$$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , \lambda > 0, x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

MGF: $M_x(t) = E(e^{tx})$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{0}^{\infty} e^{tx} \lambda e^{-\lambda x}\,dx$$

$e^{tx} e^{-\lambda x} = e^{(t-\lambda)x} = e^{-(\lambda - t)x}$

$$= \lambda \int_{0}^{\infty} e^{-(\lambda - t)x}\,dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda - t)} \right]_0^{\infty}$$

$$= \lambda \left[\frac{e^{-\infty}}{-(\lambda-t)} - \frac{e^{0}}{-(\lambda-t)} \right]$$

$$= \lambda \left(0 + \frac{1}{\lambda - t} \right)$$

$$M_x(t) = \frac{\lambda}{\lambda - t}$$

Mean $= E(x) = \left[\frac{d}{dt} M_x(t) \right]_{t=0}$

$$= \left[\frac{d}{dt} \frac{\lambda}{\lambda - t} \right]_{t=0} = \left[\frac{-\lambda}{(\lambda - t)^2} (-1) \right]_{t=0}$$

$$= \frac{\lambda}{\lambda^2}$$

Mean $= E(x) = \frac{1}{\lambda}$

$$E(x^2) = \left[\frac{d^2}{dt^2} M_x(t) \right]_{t=0} = \left[\frac{d}{dt} \frac{\lambda}{(\lambda - t)^2} \right]_{t=0}$$

$$= \left[\frac{-2\lambda}{(\lambda - t)^3} (-1) \right]_{t=0}$$

$$= \frac{2\lambda}{\lambda^3}$$

$$E(x^2) = \frac{2}{\lambda^2}$$

Variance $= E(x^2) - (E(x))^2$

$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$

$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$

Variance $= \frac{1}{\lambda^2}$

3. Buses arrive at a specific stop at 15 minutes intervals starting at 7am. That is they arrive at 7, 7.15, 7.30, 7.45 and so on. If a passenger arrives at the stop at the time is uniformly distributed between 7 and 7.30 am. Find the probability that he waits for a bus.

Sol. X is a random variable which is UD in the interval (0, 30)

$$f(x) = \begin{cases} 1/30 & , \ 0 < x < 30 \\ 0 & , \ \text{otherwise} \end{cases}$$

i) less than 5 minutes for a bus.

P[passengers will have to wait less than 5 mins]

= P[passengers may arrive b/w 7.10 to 7.15] + P[passengers may arrive b/w 7.25 to 7.30]

$$= \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx = \frac{1}{30}\left[[x]_{10}^{15} + [x]_{25}^{30}\right]$$

$$= \frac{1}{30}(5+5) = \frac{10}{30}$$

$$= \frac{1}{3}$$

ii) More than 12 minutes

P[Passengers will have to wait at least 12 mins]

= p[passengers arrive b/w 7 to 7.03] + P[passengers arrive b/w 7.15 to 7.18]

$$= \int_0^3 \frac{1}{30} f dx + \int_{15}^{18} \frac{1}{30} dx = \frac{1}{30}\left[(x)_0^3 + (x)_{15}^{18}\right]$$

$$= \frac{1}{30}(3+3) = \frac{6}{30} = \frac{1}{5}$$

Problems on Exponential Distribution

1. The mailage in which car owners get with a certain kind of radial tyre is a random variable having an exponential distribution with mean 40.000 km. find the probability that

Sol: X is exponentially distributed then

Probability distribution function is

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0, \lambda > 0.$$

Given, Mean = 40000

$$\frac{1}{\lambda} = 40000$$

$$\lambda = \frac{1}{40000}$$

$$f(x) = \frac{1}{40000} e^{-\frac{1}{40000}x}, \quad x > 0$$

i) one of the tyres will lost atleast 20,000 km.

$$P[\text{the tyre will lost atleast } 20000] = P[X \geq 20000]$$

$$= e^{-\lambda k} \qquad \because k = 20,000 \qquad P[X>k] = e^{-\lambda k}$$

$$= e^{-\frac{1}{40000}(20000)}$$

$$= e^{-1/2}$$

$$= 0.6065.$$

ii) one of the tyres will lost atmost 50000 km.

$$P[\text{the tyre will lost atmost } 50000]$$

$$= P[X \leq 50000]$$

$$= 1 - P[X > 50000]$$

$$= 1 - e^{-\frac{1}{40000} \times 50000}$$

$$= 1 - e^{-5/4}$$

$$= 1 - 0.2865$$

$$= 0.7135$$